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MV-Algebra for Cultural Rules

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Abstract This paper reports preliminary results on a new area of application of quantum structures, motivated by a reading of the 2004 monograph *Reasoning in Quantum Theory*. Ethnographers often describe a particular culture by describing rules of social relations that they assert characterize that culture. Viable cultures exist over periods of time, that is, over sequences of "generations". To embody this, we define a suitable set of objects and relations, and a structure on which cultural rules act as "operators" on a set of "configurations" on generations. This yields an MV-algebra of those operators. This implies that culture theory might be studied as an example of the theory of quantum structures.

Keywords Quantum structures · MV-algebra · GDP · Cultural theory · Operators for cultural rules · Quantum logic · Cultural rules · Mathematical anthropology

Introduction

It has been known for half a century that certain cultural rules lend themselves to operator representations [15, 17, 18, 20, 21]. But despite wide use in ethnography of the objects here called "regular structures"[1] the only predictive theory using those objects is the author's

[1] For an excellent example see Gould [11].

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previous work[2] showing that certain cultural rules can be represented by regular structures, whose properties successfully predict empirical population measures based on the structure of the rule alone. That this is a form of reasoning found in study of quantum structures became apparent upon a reading of the 2004 text *Reasoning in Quantum Theory* [9]. But to show that may be true requires a restatement of the foundations of culture theory. The present paper is the first result of that effort.[3]

1 Basic Sets and Relations

While definitions are general, the reader may imagine application of these objects to an evolving system (perhaps a particular biological species, or specific cultural group), consisting of a "population" $\mathbb{P}$, composed of individual members b, c, etc., organized into disjoint subsets $\mathbb{G}$ called "generations", in turn organized into sequences by relations called "descent" denoted "D", while within particular generations may be found certain graphs called "configurations" composed of relations "M" denoting "marriage", and "B" for "siBships" (sets of individuals in a given $\mathbb{G}$ with common parents). We write $\#\mathcal{H}$ for the number of elements in a set $\mathcal{H}$, and write ":=" to define that the symbol on the left of ":=" means the object on the right. We use the symbol B to represent a particular sibship of $\mathbb{P}$, i.e., $\mathrm{B} := \{c \in \mathbb{P} \mid c \in bB)\}$ for some fixed $b \in \mathbb{P}$. We use the symbol M to represent a particular marriage of $\mathbb{P}$, i.e., $\mathrm{M} := \{c \in \mathbb{P} \mid c \in bM)\}$ for some fixed $b \in \mathbb{P}$.

Definition 1 An *evolutionary structure* $\mathbb{S}$ is a quintuple $(\mathbb{P}, \mathbb{R}, D, B, M)$ where $\mathbb{P}$ is a non-empty set, $\mathbb{R}$ is a non-empty set, and D, B, and M are binary relations on $\mathbb{P}$ satisfying the following:

(1) D is totally non-symmetric and totally non-reflexive;
(2) If bDc and there exists no $d \in \mathbb{P}$, $d \neq b, c$ for which bDd and dDc, then c is a *parent* of b and b is an *offspring* of c;
(3) If $b, c, d \in \mathbb{P}$, and both bDd and cDd then bBc;
(4) M is transitive and symmetric;
(5) If bMc and c is a parent of d, then b is a parent of d; thus
(6) $\#bM \leq 2$ for all $b \in \mathbb{P}$.

The result of Definition 1(6) is that marriage is "monogamous" between any pair of "married" individuals. For convenience we shall denote also that if bDc and c is a parent of b, then we can also write with the same meaning, $cD^{-1}b$. Note that D imposes a partition on $\mathbb{P}$, and that bBc iff there exists a $d \in \mathbb{P}$ such that $bD^{-1}d$ and $cD^{-1}d$.

Given a subset $\mathbb{G}$ of $\mathbb{P}$, we say that a relation Z *respects* $\mathbb{G}$ in case, for all $b \in \mathbb{G}$, bZ is a subset of $\mathbb{G}$. Given a subset $\mathbb{G}$ of $\mathbb{P}$, we say that a pair of relations Z, W *splits* $\mathbb{G}$ in case, for all $b \in \mathbb{G}$, $bZ \cap bW$ is never a subset of $\mathbb{G}$.

Definition 2 Henceforth we assume that $\mathbb{S} = (\mathbb{P}, \mathbb{R}, D, B, M)$ is an evolutionary structure that has a set of (*discrete*) *generations, with the index set* $\mathcal{T}$ *of integers*

$$\Delta = \{\mathbb{G}^t \mid t \in \mathcal{T}, \mathbb{G}^t \subseteq \mathbb{P}\}$$

satisfying the following properties:

[2] Ballonoff [1–5, 6, Chap. 4, 7]. Selected texts of [6] are available on-line at http://www.BallonoffConsulting.com/?PG=publications.

[3] Examples and more extended discussions may be found in [8].

(1) M and B respect each $\mathbb{G}^t \in \Delta$;
(2) The pairs D, B and D, M split each $\mathbb{G}^t \in \Delta$;
(3) If $b, c \in \mathbb{P}$, $\mathbb{G}^t \in \Delta$, $b \in \mathbb{G}^t$, and c is a parent of b then $c \in \mathbb{G}^{t-1}$;
(4) Δ is a partition of $\mathbb{P}$, that is, $\bigcup_t \mathbb{G}^t = \mathbb{P}$ for $t \in \mathcal{T}$, and $\mathbb{G}^i \cap \mathbb{G}^t = \emptyset$ for $i \neq j$, $i, j \in \mathcal{T}$.

Let $\mathbb{S} = (\mathbb{P}, \mathbb{R}, D, B, M)$ be an evolutionary structure, with set of generations Δ, let $\mathcal{T}$ be the time index of $\mathbb{S}$, let $b, c, d \in \mathbb{P}$, let $\mathbb{G}^t \in \Delta$ be a generation of Δ, and let c be a parent of b. Then bDc. If c is a parent of b, and d is a parent of c, and if $b \in \mathbb{G}^t$, then all of bDc, bDd, $c \in \mathbb{G}^{t-1}$ and $d \in \mathbb{G}^{t-2}$. If $b \in \mathbb{G}^t$ and bBc then $c \in \mathbb{G}^t$, and if bMd then also $d \in \mathbb{G}^t$. Thus given an evolutionary structure $\mathbb{S} = (\mathbb{P}, \mathbb{R}, D, B, M)$ the set $\Delta = \{\mathbb{G}^t \mid t \in \mathcal{T}\}$ is the indexed set of generations $\mathbb{G}^t$ induced on $\mathbb{P}$ by D. We thus also refer to such a Δ as a *descent sequence of* $\mathbb{S}$, or simply a *descent sequence*.

Definition 3 Let $\mathbb{S} = (\mathbb{P}, \mathbb{R}, D, B, M)$ be an evolutionary structure with descent sequence $\Delta = \{\mathbb{G}^t \mid t \in \mathcal{T}\}$, then for $\mathbb{G}^t \in \Delta$ define the sets:

$\mathbb{M}^t := \{\mathrm{M} \mid \mathrm{M} \subseteq \mathbb{G}^t\}$ the set of all marriages in the tth generation of $\mathbb{S}$;
$\mathbb{B}^t := \{\mathrm{B} \mid \mathrm{B} \subseteq \mathbb{G}^t\}$ the set of all sibships in the tth generation of $\mathbb{S}$;
$\mathbb{M} := \bigcup \mathbb{M}^t$, over $t \in \mathcal{T}$, the set of all marriages in the evolutionary structure $\mathbb{S}$;
$\mathbb{B} := \bigcup \mathbb{B}^t$, over $t \in \mathcal{T}$, the set of all sibships in the evolutionary structure $\mathbb{S}$.

Definition 4 Let $\mathbb{S} = (\mathbb{P}, \mathbb{R}, D, B, M)$ be an evolutionary structure, let $\Delta = \{\ldots, \mathbb{G}^{t-1}, \mathbb{G}^t, \mathbb{G}^{t+1}, \ldots\}$ be a descent sequence of $\mathbb{S}$. Let $\mathbb{M}^t$ and $\mathbb{B}^t$ be sets defined on Δ as in Definition 3. Then for $\mathbb{G}^t \in \Delta$ let $\gamma^t := \#\mathbb{G}^t$, and let $\beta^t := \#\mathbb{B}^t$, and let $\mu^t := \#\mathbb{M}^t$.

The concepts "split" and "respect" induce a *generational coherence of cells*: they assure that the cells B each occur in only one generation. That the subsets M each occur in only one generation, and that if $\mathrm{M} \subset \mathbb{P}$, $\mathrm{B} \subset \mathbb{P}$ and M contains the parents of the individuals in B, then the members of M and B are not in the same generation.

2 Forward Sequences and Subsequences

For each evolutionary structure $\mathbb{S}$, Definition 1 does not require that there necessarily exists a $c \in \mathbb{P}$ such that bDc. If we require $\forall b \in \mathbb{P}\ \exists c \in \mathbb{P}$ and bDc, then if $t \in \mathcal{T}$, then necessarily $\mathcal{T}$ is infinite in the "backward" direction. While life has existed for a very large number of generations, it's history is not, to present knowledge, infinite in the backward direction, and is only conditionally infinite in the forward direction; thus see Definition 5(1) and 5(2), and the discussion of "viable" sequences in Sect. 7 below. Also, we can imagine that a descent sequence is itself composed of "parallel" sets of non-interacting "(sub)sequences" for at least finite numbers of generations, motivating below Definition 5(3) through 5(5).

Definition 5 Let $\mathbb{S} = (\mathbb{P}, \mathbb{R}, D, B, M)$ be an evolutionary structure, with descent sequence Δ, time index set $\mathcal{T}$, and let $b \in \mathbb{P}$.

(1) If there exists a $t \in \mathcal{T}$ such that there exists a non-empty $\mathbb{G}^t \in \Delta$ but $\mathbb{G}^{t-1} \notin \Delta$ then such D is a *forward descent sequence of* $\mathbb{S}$, or simply a forward descent sequence, if $\mathbb{S}$ is understood.
(2) If for every $t \in \mathcal{T}$ there exists a non-empty $\mathbb{G}^t \in \Delta$ for which also $\mathbb{G}^{t-1} \in \Delta$ then such Δ is a *complete descent sequence* of $\mathbb{S}$.

(3) Let Δ_{G} and Δ_{H} be forward descent sequences of $\mathbb{S}$, and let $\Delta_{\mathrm{G}} \cap \Delta_{\mathrm{H}} = \emptyset$. Then Δ_{G} and Δ_{H} are called *independent descent sequences* of $\mathbb{S}$, and Δ_{G} is said to be *independent of* Δ_{H}.
(4) Let $\Delta_{\mathrm{G}} = \{\mathbb{G}^t \mid t \in \mathcal{T}_{\mathrm{G}}\}$ be a (forward or complete) descent sequence of $\mathbb{S}$. Let $\Delta_{\mathrm{H}} = \{\mathbb{H}^t \mid t \in \mathcal{T}_{\mathrm{H}}\}$ be a forward descent sequence of $\mathbb{S}$, let $\mathbb{P}^* := \bigcup_t \mathbb{G}^t$, $t \in \mathcal{T}_{\mathrm{G}}$. If for every $\mathbb{H}^t \in \Delta_{\mathrm{H}}$ and every $b \in \mathbb{H}^t$ both $b \in \mathbb{P}^*$ and there exists a $c \in \mathbb{P}^*$ such that bDc, then Δ_{H} is a *subsequence* of Δ_{G}.
(5) Let $\Delta_{\mathrm{G}} = \{\mathbb{G}^t \mid t \in \mathcal{T}_{\mathrm{G}}\}$ be a (forward or complete) descent sequence of $\mathbb{S}$, let $\Delta_{\mathrm{H}} = \{\mathbb{H}^t \mid t \in \mathcal{T}_{\mathrm{H}}\}$ let $\Delta_{\mathrm{J}} = \{\mathbb{J}^t \mid t \in \mathcal{T}_{\mathrm{J}}, \mathbb{J}^t \subset \mathbb{P}\}$ be disjoint subsequences of Δ_{G}. Then Δ_{H} and Δ_{G} are *independent subsequences* of $\mathbb{S}$. The sets $\mathcal{T}_{\mathrm{G}}$ and $\mathcal{T}_{\mathrm{H}}$ are called the *local time index* of Δ_{G} and Δ_{H} respectively.

In the below, when we use the phrase "descent sequence" with no other qualifier, it means "forward descent sequence". Each evolutionary structure $\mathbb{S}$ has only one descent sequence Δ. Thus, in adding independent descent (sub)sequences we are adding disjoint subsets of the generations of a common descent sequence of an evolutionary structure $\mathbb{S}$. Note that any subset of a particular generation $\mathbb{G}^t$ is itself an independent descent subsequence, relative to the rest of that particular generation $\mathbb{G}^t$; this enables us to perform addition of such subsets, and of the configurations on them.

An evolutionary structure $\mathbb{S}$ has a *complete genealogy* if for every $b, c \in \mathbb{P}$ it is possible to determine whether bBc is true or not true. For this paper we assume all evolutionary structures $\mathbb{S}$ have a complete genealogy.[4] Note that since the cells of $\mathcal{B}$ are disjoint, that is, since each $b \in \mathbb{P}$ belongs to only one $\mathcal{B} \in \mathbb{B}$ then the sets $\mathcal{B} \in \mathbb{B}$ partition $\mathbb{P}$. Since B respects $\mathbb{P}$, then B splits $\mathbb{P}$, and if bBc and $b \in \mathbb{G}^t \subseteq \mathbb{P}$, then also $c \in \mathbb{G}^t$. M also respects and thus splits $\mathbb{P}$, but, despite that the sets $\mathcal{M}$ are disjoint, the sets $\mathcal{M} \in \mathbb{M}$ partition $\mathbb{P}$ iff every $b \in \mathbb{P}$ has at least one offspring. This may occur but nothing in the axiomatic structure assures it. Thus see also discussion of "viable" sequences in Sect. 7.

3 Descent Map Definition

Definition 6 Let $\mathbb{S} = (\mathbb{P}, \mathbb{R}, D, B, M)$ be an evolutionary structure with time index set $\mathcal{T}$ and descent sequence Δ. Then let

$$\mathbb{D} : \mathbb{B} \to \mathbb{M}$$

be the map from the subsets $\mathcal{B} \in \mathbb{B}$ of sibships of $\mathbb{P}$ onto the reproducing subsets $\mathcal{M} \in \mathbb{M}$ of individuals in $\mathbb{P}$, associating with each sibship $\mathcal{B} \in \mathbb{B}$ the set $\mathcal{M} \in \mathbb{M}$ that is ascribed as the parents of the individuals $b \in \mathcal{B}$. Call $\mathbb{D}$ the *descent map* ("descendant of" map) on $\mathbb{P}$.

Let $\mathbb{G}^t, \mathbb{G}^{t-1} \in \Delta$, let $\mathcal{B}^t$ be the set of all sibships $\mathcal{B} \in \mathbb{G}^t$, and let $\mathcal{M}^{t-1}$ be the set of all reproducing marriages $\mathcal{M} \in \mathbb{G}^{t-1}$. The mapping $\mathbb{D}^t : \mathcal{B}^t \to \mathcal{M}^{t-1}$ is onto since $\mathcal{M}^{t-1}$ is the

[4] This implies that there may be $b \in \mathbb{P}$ and $d \notin \mathbb{P}$ for which bDd. That is, the objects in the population under study may have initially evolved from some other object, not in that population. By the current definition we avoid the necessity to resolve issues of fundamental cosmology and biological evolution in the present paper. Eventually culture theory shall have to deal explicitly with issues of origin. Thus, it is probably more than coincidental that the notion "causal set" as used in the literature on cosmology, imposes a partial order similar to our descent sequences, and like here, deals with non-reproductive objects by defining them as "bystanders" and omitting them from certain statistics. See also footnote 7 below.

set of all sets of parents that have at least one descendant (that it, we ignore non-reproducing sets $\mathcal{M}$.

Lemma 1 *Let* $\mathbb{S} = (\mathbb{P}, \mathbb{R}, D, B, M)$ *be an evolutionary structure with descent sequence* Δ, *and let* $\mathbb{D}:\mathbb{B} \to \mathbb{M}$ *be the descent map on* Δ. *Then the density function of possible sizes of the sets* $\mathcal{B} \in \mathcal{B}^t$ *onto the sets* $\mathcal{M} \in \mathcal{M}^{t-1}$, *and thus also of sizes of the sets* $\mathcal{B} \in \mathcal{B}^t$ *onto* $\mathcal{M} \in \mathcal{M}^{t-1}$, *is given by the Stirling Number of the Second Kind.*

Proof Because $\mathbb{D}$ is a surjection (from sets $\mathcal{B} \in \mathbb{B}^t$ onto $\mathcal{M} \in \mathbb{M}^{t-1}$), this proof is found in many texts in the standard literature on combinatorics of surjections.[5] □

Lemma 1 is an important result. It allows us to compute, *inter alia*, a density function on the possible sizes of the sets $\mathcal{B} \in \mathbb{B}^t$ of a given generation, given the number of reproducing sets $\mathcal{M} \in \mathbb{M}^{t-1}$ of the previous generation, which number in turn may depend on the marriage rule. That is, given certain purely "structural" knowledge, we can compute a predicted numerical value of an "observable", such as "average family size". Therefore, our seemingly simple model implies a very strongly predictive tool for empirical measures on populations, determined by particular rules. This theory is laid out in Ballonoff [1, 3, 4, 8].

Axiom *We require that* $\mathbb{D}$ preserves *the relations* D, *that is,* $\mathbb{D}(\mathcal{B}) = \mathcal{M}$ *preserves* D *iff for* $b, c \in \mathbb{P}$, *if* $cD^{-1}b$, $b \in \mathcal{B} \in \mathbb{B}$ *then* $c \in \mathcal{M} \in \mathbb{M}$. *And thus also for given* $t \in \mathcal{T}$, $\mathbb{D}(\mathcal{B}) = \mathcal{M}$ *iff* $b, c \in \mathbb{P}$, $\mathbb{G}^t, \mathbb{G}^{t-1} \in \Delta$, $b \in \mathbb{G}^t$, $c \in \mathbb{G}^{t-1}$, $cD^{-1}b$, $b \in \mathcal{B} \in \mathbb{B}^t$ *and* $c \in \mathcal{M} \in \mathbb{M}^{t-1}$.

Therefore $\mathbb{D}$ simply collects all of the relationships D between two generations, and maps them all simultaneously. Because in an evolutionary structure with a complete genealogy, the sets $\mathcal{B} \in \mathbb{B}$ partition $\mathbb{P}$, it is also true that each reproducing set $\mathcal{M} \in \mathbb{M}$ has mapped onto it exactly one $\mathcal{B} \in \mathbb{B}$, so D is 1–1. So we can create an inverse map $\mathbb{D}^{-1}:\mathbb{M} \to \mathbb{B}$ called the *ancestor* map ("ancestor of" map) on $\mathbb{P}$, which we also require to preserve D, and thus which is also 1–1 and onto $\mathbb{B}$. Therefore also all of $\mathbb{D}$, $\mathbb{D}^{-1}$ and their specific forms $\mathbb{D}^t$ and $(\mathbb{D}^t)^{-1}$, are bijections.

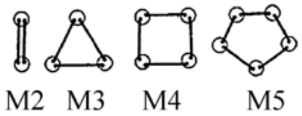
4 Vectors of Configurations

This section provides an intuitive description of configurations on generations and the set of vectors representing configurations. A general algebraic treatment is warranted, but we can go quite far here using the simple "regular structures", described below, which are easily illustrated by examples. The term *concrete configuration* shall mean the pair $\mathrm{C}_t = (\mathbb{B}^t, \mathbb{M}^t)$ consisting of the partition $\mathbb{B}^t$ and the sets $\mathbb{M}^t$ on $\mathbb{G}^t$. Thus, the $\mathbb{B}^t$ and $\mathbb{M}^t$ also identify sets of relationships or graphs on the individuals in $\mathbb{G}^t$. Because the elements of $\mathbb{M}^t$ and $\mathbb{B}^t$ are just subsets of $\mathbb{G}^t$, for any two generations $\mathbb{G}^i, \mathbb{G}^j \in \Delta$, we can also study whether the graph of $\mathbb{M}^i$ and $\mathbb{B}^i$ on $\mathbb{G}^i$ is isomorphic to that of $\mathbb{M}^j$ and $\mathbb{B}^j$ on $\mathbb{G}^j$.[6]

[5] For proofs that a surjection requires the Stirling Number of the Second Kind as its density function, see for example van Lint, J.H. and R.M. Wilson [19], p. 106; Grimaldi, R.P. [12], p. 178; and Peter Hilton, Jean Peterson and Jurgen Stiger [14].

[6] See for example DeMeur and Gottscheiner [10].

We define a notation for certain isomorphism classes called *regular structures*. To depict these use a dot to represent an individual, use a circle around two dots (say, b and c) to show that bMc, and use a line between pairs two dots (say, d, e) to show that they are of sibs in the same sibship dBe. When these form simple closed cycles "sibship-marriage-sibship-marriage ..." closing back to the first listed sibship, we can give them names M1, M2, M3, etc. according to the number of $\mathcal{M}$ subsets in each figure, and identifying only closed (cyclic) figures, of disjoint $\mathcal{B}$-sets and disjoint $\mathcal{M}$-sets. For example:



are the regular structures with 2, 3, 4 and 5 M-sets, represented by the circles.[7]

Using these structures, if they are all that is present, we can represent the isomorphism classes of a particular concrete configuration C^t as a *configuration* vector

$$\mathrm{C} = (m_0, m_1, m_2, \ldots, m_j, \ldots)$$

where the coefficient m_j is the number of elements of isomorphism type M_j in the concrete configuration. We note that for all j, $m_j \in R^+$, the set of non-negative real numbers; more specifically in the present paper all $m_j \in N^+$, the non-negative integers.[8]

Note that each C is an n-tuple describing the isomorphism class of a concrete configuration $\mathrm{C}_t = (\mathbb{B}^t, \mathbb{M}^t)$ on some subset $\mathbb{G}^t$ of $\mathbb{P}$. When we mean the concrete configuration we shall use the form C_t. The ordered sequence consisting of just the coefficients m_j is sufficient to uniquely describe any particular isomorphism class represented by such "vector". The configuration consisting of no relations is the vector $(0, 0, \ldots)$. When no ambiguity arises we denote this vector as "(0)" or simply "0".

5 Algebraic Operations on Configurations

Assume a non-empty descent sequence $\Delta = \{\ldots, \mathbb{G}^{t-1}, \mathbb{G}^t, \mathbb{G}^{t+1}, \ldots\}$ of an evolutionary structure $\mathbb{S}$. We now discuss addition of two independent subsequences, whose union in a particular generation is $\mathbb{G}^t$. That is, for disjoint subsets $\mathbb{G}', \mathbb{G}'' \subset \mathbb{G}^t$ we will define addition of configurations to be the result of the union $\mathbb{G}' \cup \mathbb{G}''$. Thus $\mathbb{G}' + \mathbb{G}''$ means a set that takes the union of all of the individuals of the two sets, and also preserves all of the relations on $\mathbb{G}'$ and on $\mathbb{G}''$. So on the set $\mathbb{G}' \cup \mathbb{G}''$ of disjoint sets $\mathbb{G}'$ and $\mathbb{G}''$ we also find all of the configurational elements on $\mathbb{G}'$ as well as on $\mathbb{G}''$.

As a numerical example, if $\mathrm{C}' + \mathrm{C}''$ are on disjoint subsets $\mathbb{G}'$ and $\mathbb{G}''$ of a generation $\mathbb{G}^t$, and $\mathrm{C}' = (0, 0, 2(\mathrm{M2}), 0, 0, \ldots)$ and $\mathrm{C}'' = (0, 0, 3(\mathrm{M2}), 0, 0, \ldots)$, then the result of $\mathrm{C}' + \mathrm{C}'' = (0, 0, 5(\mathrm{M2}), 0, 0, \ldots)$. Without listing the basis objects, we have:

$$\mathrm{C}' + \mathrm{C}'' = (0, 0, 2, 0, 0, \ldots) + (0, 0, 3, 0, 0, \ldots) = (0, 0, 5, 0, 0, \ldots).$$

[7] Not illustrated and not otherwise used in this paper, is the M0 configuration, which would be a dot with a circle around it. Such configuration could represent for example, a single cellular organism that requires no "partner" to reproduce. As that possibility will also be exploited in other papers, we leave the vector position for the M0 configuration as the first listed position in our vectors, for completeness. M1 would be a "marriage" of two individuals, shown as two dots in a circle with no other structure illustrated.

[8] We prefer to use R^+ since in future papers we expect to develop a "normalization" of certain of the sets below, which will require real coefficients other than integers.

Thus let $\mathrm{C}, \mathrm{D}, \mathrm{E} \in \mathbb{C}$ be configurations respectively on disjoint non-empty $\mathbb{G}', \mathbb{G}'', \mathbb{G}''' \subseteq \mathbb{G}^t$ of a given descent sequence Δ. Since for all j, $m_j \in R^+$, with respect to arithmetic addition the set $\mathbb{C}$ of isomorphism classes of configurations on disjoint subsets of $\mathbb{G}^t$ thus is: *closed*, since if $\mathrm{C}, \mathrm{D} \in \mathbb{C}$, then $\mathrm{C}+\mathrm{D} \in \mathbb{C}$, by ordinary arithmetic addition; *commutative:* since ordinary arithmetic addition of non-negative numbers commutes, thus $\mathrm{C}+\mathrm{D} = \mathrm{D}+\mathrm{C}$; *associative*: since by ordinary arithmetic addition at each coefficient $(\mathrm{C}+\mathrm{D})+\mathrm{E} = \mathrm{C}+(\mathrm{D}+\mathrm{E})$; and an *additive identity* exists—denote by 0 the vector $(0, 0, 0, \ldots)$ in which $m_j = 0$ for all j so $\mathrm{C}+0 = 0+\mathrm{C} = \mathrm{C}$. Note, $\mathbb{C}$ is also closed under scalar multiplication by the integers.[9] Let $nm_i = m_i + m_i + \cdots$, performed n times. Then if $\mathrm{C} \in \mathbb{C}$ is multiplied by a scalar n; $n\mathrm{C} \in \mathbb{C}$. And if $\mathrm{C}, \mathrm{D} \in \mathbb{C}$ then $n(\mathrm{C}+\mathrm{D}) = (n\mathrm{C} + n\mathrm{D}) = n\mathrm{C} + n\mathrm{D}$ which is just addition of vectors in $\mathbb{C}$ so the result is also in $\mathbb{C}$. Denote the algebra just defined on the set $\mathbb{C}$ with operation "+" and zero element 0, $(\mathbb{C}, +, 0)$.

Now we ask, is there an operation "$-$" such that for any $\mathrm{C} \in \mathbb{C}$, $\mathrm{C} - \mathrm{C} = 0$, and such that the operation $\mathrm{C} - \mathrm{D}$ is defined even when $\mathrm{C} \neq \mathrm{D}$. Since "subtraction" of configurations "removes" part of the structure on a generation, when we "subtract" C' from C we can only take from C as much as is in it. Thus if, at some j, $m_j < m'_j$ then $m_j - m'_j \geq 0$. For example at position $j = 2$ if C has 2 of the Mj configurational elements, then $m_j = 2$. And if C' has 3 of the Mj elements then $m'_j = 3$. Thus $m_j - m'_j = 2 - 3 = 0$, not $2 - 3 = -1$. However, if we instead subtract $\mathrm{C}' - \mathrm{C}$ then at the j position $m_j \circ m'_j = \max(0, m_j - m'_j)$, we have $m'_j - m_j = 3 - 2 = 1$. Thus following Hedlikova and Pulmannova [13], Definitions 1.1 and 2.1:

Definition 7 Let $(X, \leq)$ be a poset, with a smallest element 0, on which exists a partial binary operation $\circ$ such that for $x, y \in X$ if $x \circ y$ is defined, then $x \circ y$ is called a *difference* on X iff, for $z, y, z \in X$ the following are satisfied:

(D1) if $x \leq y$ then $y \circ x \leq y$ and $y \circ (y \circ x) = x$;
(D2) if $x \leq y \leq z$ then $z \circ y \leq z \circ x$ and $(z \circ x) \circ (z \circ y) = y \circ x$; and
(C) if $x \leq y, z$ and $y \circ x = z \circ x$ then $y = z$.

The system $(X, \leq, 0, \circ)$ is called a *generalized difference poset* (GDP).

Definition 8 Let $\mathbb{S}$ be an evolutionary structure with descent sequence Δ. Let $\mathbb{G}^t \in \Delta$, let $\mathbb{G}$ and $\mathbb{G}'$ be disjoint subsets of $\mathbb{G}^t$. Let C and C' be the configurations on $\mathbb{G}$ and $\mathbb{G}'$ respectively. Let the coefficients of C and C' at position j be m_j and m'_j, and let a, b and c be possible values of m_j and m'_j. For each j, define the form $\mathrm{C} \circ \mathrm{C}'$ to be value at the jth coefficient $m_j \circ m'_j = \max(0, m_j - m'_j)$ such that if $a + b = c$, then necessarily all of: (i) $a \leq c$; (ii) $b \leq c$; and (iii) if $b \geq a$ then $a \circ b = 0$.

Lemma 2 *Let $\mathbb{C}$ be the set of configurations, and let $\leq$ and $\circ$ be the operation defined in Definition 8. Then the system $(\mathbb{C}, \leq, 0, \circ)$ is a GDP.*

Proof We know $0 \in \mathbb{C}$. For $\mathrm{C} \in \mathbb{C}$ at all positions j, $m_j \in R^+$, and from Hedlikova and Pulmannova [13], the set R^+ with subtraction $x \circ y' = \max(0, x - y)$ is a GDP. It is evident that the operation $\leq$ of Definition 8 forms a partial order. □

Write a GDP thus defined as $(\mathbb{C}, -, 0)$, or $(\mathbb{C}_j, -, 0)$ for just the jth position of the vector corresponding to regular structure Mj.

[9] Because we are for now dealing only with regular structures, and finite generations, it is sufficient that the scalar multiplication be integers.

6 Properties of the Rule Algebra

A rule is any statement in natural, logical or mathematical language which determines how a relationship on a given generation may form. For example, a marriage rule is a statement that says how the M relation may, or may not, form. Even when a rule is not explicitly stated in terns of configurations, it may and often will affect how configurations may form. For example, the rule "one may not marry themselves, nor a person who is a sibling of themselves, but must marry within the same generation", means that M2 is the smallest structure which can "reproduce itself" in one generation. Thus while reserving for other papers a more general treatment of rules, we here concentrate on the effects of rules on formation of configurations; that is, we study rules as "operators" on $\mathbb{C}$.

Definition 13 Let $\mathbb{S} = (\mathbb{P}, \mathbb{R}, D, B, M)$ be an evolutionary structure with time index set $\mathcal{T}$ and descent sequence $\Delta = \{\mathbb{G}^t \mid t \in \mathcal{T}\}$. Let $\mathrm{C}, \mathrm{D} \in \mathbb{C}$. Let $\mathrm{R} \in \mathbb{R}$ be a rule and let $\mathbf{R} : \mathbb{C} \to \mathbb{C}$ be the rule operator for R on $\mathbb{C} \times \mathbb{C}$. Let $\mathsf{R} := \{\mathbf{R} \mid \mathrm{R} \in \mathbb{R}, \mathbf{R}$ is a rule operator for R$\}$ be the set of rule operators of $\mathbb{S}$.

(1) Let:
  (i) $\mathrm{R}_{\mathrm{C,D}} = 1$ iff $\mathbf{R}$ allows (or requires) a transition from C to D.
  (ii) $\mathrm{R}_{\mathrm{C,D}} = 0$ iff $\mathbf{R}$ does not allow a transition from C to D.

(2) Let

$$\mathbf{R}\mathrm{C} := \{\mathrm{D} \mid \mathrm{D} \in \mathbb{C} \text{ and } \mathrm{R}_{\mathrm{C,D}} = 1\}$$

and call $\mathbf{R}$C the *set of accessible configurations* under R starting from C.

(3) Let

$$\mathbb{C}^{\mathbf{R}} := \{\mathbf{R}\mathrm{C} \mid \mathrm{R} \in \mathbb{R}, \mathbf{R} \in \mathsf{R} \text{ the operators for } \mathrm{R} \in \mathbb{R}, \mathrm{C} \in \mathbb{C}\}$$

be the set of sets of accessible configurations under any $\mathrm{R} \in \mathbb{R}$ starting from any $\mathrm{C} \in \mathbb{C}$.

Write $\mathbf{R}^2\mathrm{C} = \mathbf{R}(\mathbf{R}\mathrm{C})$ to mean the result of application of the operator $\mathbf{R}$ for the rule R for two successive generations starting from the configuration C, that is $\mathbf{R}^2\mathrm{C} = \mathbf{RR}\mathrm{C}$. And generically $\mathbf{R}^k\mathrm{C}$ for the result of application of R for k successive steps starting from C. Write $\mathbf{RS}\mathrm{C} = \mathbf{R}(\mathbf{S}\mathrm{C})$ for the application of rule S to C followed by the application of rule R to the result, which then also allows forms like $\mathbf{R}^k\mathbf{S}^j\mathrm{C}$, with the obvious meaning. We shall call the form "$\mathbf{RS}$" the "sequential application" of rules. Only in particular cases is $\mathbf{R}$C a unique outcome. In such case we write $\mathbf{R}\mathrm{C} = \mathrm{D}$.

Following Definition 13 we can also compute the set $\mathbf{R}(\mathbf{S}\mathrm{C}) = \mathbf{RS}\mathrm{C}$ of possible outcomes of sequential application of rule S to configuration C followed by rule R as follows.

Definition 14 Under the same premises as Definition 13,

$$\mathbf{RS}\mathrm{C} := \bigcup_{\mathrm{D}} \mathbf{S}\mathrm{D}, \quad \mathrm{D} \in \mathbf{R}\mathrm{C}.$$

That is, for $\mathrm{D}, \mathrm{F}, \ldots \in \mathbf{R}\mathrm{C}$, then $\mathbf{SR}\mathrm{C} = \mathbf{S}\mathrm{D} \cup \mathbf{S}\mathrm{F} \cup \cdots$.

Definition 15 Let $\mathbb{R}$ be a set of rules, let $\mathrm{R}, \mathrm{S} \in \mathbb{R}$, let $\mathsf{R} = \{\mathbf{R} \mid \mathrm{R} \in \mathbb{R}$ and $\mathbf{R}$ is the operator for R$\}$ be the set of rule operators for $\mathbb{R}$, let $\mathbb{C}$ be a set of configurations, and let $\mathrm{C} \in \mathbb{C}$.

Let $\mathbf{R}\mathrm{C}$ be the set of configurations accessible from C under R, and let $\mathbf{S}\mathrm{C}$ be the set of accessible configurations under S. Let $\leq$ be a relation such that:

$$\mathbf{R}\mathrm{C} \leq \mathbf{S}\mathrm{C} \quad \text{iff} \quad \mathbf{R}\mathrm{C} \subseteq \mathbf{S}\mathrm{C}$$

Lemma 3 *Let $\leq$ be the relation defined in Definition 15. Let $\mathbb{C}^{\mathbf{R}}$ be the set of all sets of accessible configurations per Definition* 13.2. *Then* $(\mathbb{C}^{\mathbf{R}}, \leq, \mathbf{0}, \mathbf{1})$ *where* $\mathbf{1} := \{\mathrm{C} \mid \mathrm{C} \in \mathbb{C}\}$ *and* $\mathbf{0} := \emptyset$ *is a bounded poset.*

Proof Since $\leq$ is a partial order, then $(\mathbb{C}^{\mathbf{R}}, \leq)$ is a poset. Let $\mathrm{C} \in \mathbb{C}$ and let $\mathrm{R} \in \mathbb{R}$ be a rule that allows no transitions following C. Then $\mathbb{C}^{t}_{\mathrm{RC}} = \emptyset$ so $\emptyset = \mathbf{0} \in \mathbb{C}^{\mathbf{R}}$, and clearly for all $x \in \mathbb{C}^{\mathbf{R}}$, $\emptyset \leq x$; and let $\mathrm{S} \in \mathbb{R}$ be a rule that admits any configuration to follow C, so $\mathbf{R}\mathrm{C} = \mathbb{C} = \mathbf{1}$, and since any set $\mathbf{R}\mathrm{C} \in \mathbb{C}^{\mathbf{R}} \subseteq \mathbb{C}$ then for all $\mathbf{R}\mathrm{C}$, $\mathbf{R}\mathrm{C} \leq \mathbb{C}$ and we have a bounded poset. □

Definition 16 Let $\mathbb{C}$ be a finite set of configurations, including the configuration $\mathbf{0}$. Let $\mathrm{C}, \mathrm{D} \in \mathbb{C}$ and then define $a_{\mathrm{C,D}}$ such that $a_{\mathrm{C,D}} = 1$ iff $\mathrm{R}_{\mathrm{C,D}} = 1$, and $a_{\mathrm{C,D}} = 0$ iff $\mathrm{R}_{\mathrm{C,D}} = 0$. Let A_{RC} be an n-tuple that lists the 0 or 1 values of $a_{\mathrm{C,D}}$ given R for all $\mathrm{D} \in \mathbb{C}$. Let $\mathbb{A} = \{\mathrm{A}_{\mathrm{RC}} \mid \mathrm{C} \in \mathbb{C}, \mathrm{R} \in \mathbb{R}\}$ be the set of all such n-tuples. If $\mathrm{A}_{\mathrm{RC}} \in \mathbb{A}$ call A_{RC} an *accessibility indicator*.

We assume that an arbitrary "standard order" is fixed for the sequence of listing entries in an A_{RC} n-tuple. Then for every configuration $\mathrm{C} \in \mathbb{C}$ and for every rule $\mathrm{R} \in \mathbb{R}$ there is a 1–1 correspondence between $\mathbf{R}\mathrm{C}$ and each A_{RC}. Let A_{RC} be the accessibility indicator corresponding to the set $\mathbf{R}\mathrm{C}$ such that $a_{\mathrm{C,D}} = 1$ iff $\mathrm{R}_{\mathrm{C,D}} = 1$ and $a_{\mathrm{C,D}} = 0$ iff $\mathrm{R}_{\mathrm{C,D}} = 0$.

Lemma 4 *Let $\mathbb{C}$ be a set of configurations and let* $\mathrm{C}, \mathrm{D}, \mathrm{E} \in \mathbb{C}$. *Let* R *be a rule acting on* $\mathbb{C}$. *Let* $\mathrm{A}_{\mathrm{RC}}, \mathrm{A}_{\mathrm{RE}} \in \mathbb{A}$. *Define*

$$\mathrm{A}_{\mathrm{RC}} \leq \mathrm{A}_{\mathrm{RE}} \quad \textit{iff} \quad \textit{for all } \mathrm{D} \in \mathbb{C}, \quad a_{\mathrm{C,D}} \leq a_{\mathrm{E,D}}$$

Then

$$\mathbf{R}\mathrm{C} \leq \mathbf{S}\mathrm{C} \quad \textit{iff} \quad \mathrm{A}_{\mathrm{RC}} \leq \mathrm{A}_{\mathrm{SC}}.$$

The set $\mathbb{A}$ together with $\leq$ has a "maximal" accessibility indicator $\mathbf{1} = (1, 1, 1, \ldots)$ *corresponding to* $\mathbf{R}\mathrm{C} = \mathbb{C}$ *such that for all* $\mathrm{A}_{\mathrm{RC}} \in \mathbb{A}$, *for all* $\mathrm{C} \in \mathbb{C}$, $\mathrm{A}_{\mathrm{RC}} \leq \mathbf{1}$, *and there is a "minimal" accessibility indicator* $\mathbf{0} = (0, 0, 0, \ldots)$ *corresponding to* $\mathbf{R}\mathrm{C} = \emptyset$ *such that for all* $\mathrm{A}_{\mathrm{RC}} \in \mathbb{A}$, $\mathbf{0} \leq \mathrm{A}_{\mathrm{RC}}$, *and both* $\mathbf{0}, \mathbf{1} \in \mathbb{A}$. *Thus*, $(\mathbb{A}, \leq, \mathbf{0}, \mathbf{1})$ *is a bounded poset.*

Proof Obvious given the previous Lemma 3. □

Definition 17 Call a bounded poset $(\mathbb{A}, \leq, \mathbf{0}, \mathbf{1})$ constructed as in Lemma 4 an *accessibility structure*, also denoted simply as $\mathbb{A}$.

We now have a choice of paths of development. Following Sect. 5 we could proceed to show that since the sets $\mathbf{R}\mathrm{C}$ are composed of particular objects, that we can define a partial binary operation of "subtraction" among them and derive a GDP of those sets under action of differences of operators. This would lead to application of results in Hedlikova and Pulmannova [13]. However, here we emphasize the presence in culture theory of four operations on rules, namely, simultaneous application, sequential application, complementation, and "subtraction" or removal of conditions. Thus below we first derive a BCK-algebra for "substraction", and from this derive the remaining three as an MV-algebra.

Definition 18 Let $\mathbb{C}$ be a set of configurations, Define *subtraction* of two rules $\mathrm{R}, \mathrm{S} \in \mathbb{R}$ as:

$$(\mathrm{R} \circ \mathrm{S})_{\mathrm{C,D}} = 1 \quad \text{iff} \quad \text{for } \mathrm{D} \in \mathbb{C},\ \mathrm{R}_{\mathrm{C,D}} \in \mathbf{R}\mathrm{C} \quad \text{and} \quad \mathrm{S}_{\mathrm{C,D}} \notin \mathbf{S}\mathrm{C} \quad \text{else } (\mathrm{R} \circ \mathrm{S})_{\mathrm{C,D}} = 0.$$

Let $\mathbf{R}$ and $\mathbf{S}$ be the operators for rules $\mathrm{R}, \mathrm{S} \in \mathrm{R}$, and let $\mathrm{C} \in \mathbb{C}$. Define subtraction $\mathbf{R} - \mathbf{S}$ of rule operators as:

$$\mathbf{R} \circ \mathbf{S} := (\mathbf{R} - \mathbf{S})\mathrm{C} := \{\mathrm{D} \mid \mathrm{R}, \mathrm{S} \in \mathbb{R}, \mathrm{C}, \mathrm{D} \in \mathbb{C}, (\mathrm{R} \circ \mathrm{S})_{\mathrm{C,D}} = 1\}.$$

Definition 19 A BCK-algebra[10] is an algebra on a non-empty set $\mathcal{A}$ with an operation * and an element 0, such that for any $x, y, z \in \mathcal{A}$ then:

I. $((x^*y)^*(x^*z))^*(z^*y) = 0$.
II. $(x^*(x^*y))^*y = 0$.
III. $x^*x = 0$.
IV. $0^*x = 0$.
V. $x^*y = 0$ and $y^*x = 0$ imply that $x = y$.

Theorem 1 *Let $\mathbb{R}$ be a set of rules containing the empty rule, and let $\mathrm{R} \in \mathbb{R}$. Then $(\mathbb{C}^{\mathbf{R}}, \circ, \mathbf{0})$ when $\mathbf{0} = \emptyset$, is a BCK-algebra.*

Proof Obvious, since the definition of $\circ$ is as the set theoretical difference. □

Thus with respect to "subtraction" $\circ$, $\mathbb{R}$ induces a BCK-algebra $(\mathbb{C}^{\mathbf{R}}, \circ, \mathbf{0})$ of rule operators on the basis of the GDP $(\mathbb{C}, -, 0)$ of configuration vectors.

Note that it is possible, indeed likely in practice, that rules can be stated as simultaneous application of a set of independent rules. For example, the common marriage rule "one may not marry anyone who is a first cousin or closer relative" but also "one must marry within the same generation". Thus it is useful to study simultaneous application of independently stated rules.

Definition 20 Define *simultaneous application* of rules $\mathrm{R}, \mathrm{S} \in \mathbb{R}$ to be

$$(\mathrm{R} \odot \mathrm{S})_{\mathrm{C,D}} = 1 \quad \text{iff} \quad \text{for } \mathrm{D} \in \mathbb{C},\ \mathrm{R}_{\mathrm{C,D}} = 1 \quad \text{and} \quad \mathrm{S}_{\mathrm{C,D}} = 1 \quad \text{else } (\mathrm{R} \odot \mathrm{S})_{\mathrm{C,D}} = 0.$$

Let $\mathbf{R}$ and $\mathbf{S}$ be the operators for rules $\mathrm{R}, \mathrm{S} \in \mathbb{R}$, and let $\mathrm{C} \in \mathbb{C}$. Then define:

$$\mathbf{R} \odot \mathbf{S} = \{\mathrm{D} \mid \mathrm{R}, \mathrm{S} \in \mathbb{R}, \mathrm{C}, \mathrm{D} \in \mathbb{C}, \text{ and } (\mathrm{R} \odot \mathrm{S})_{\mathrm{C,D}} = 1\}.$$

Note therefore that $\mathbf{R} \odot \mathbf{R} = \mathbf{R}$, and $\mathbf{R} \odot \mathbf{S} = \mathbf{S} \odot \mathbf{R}$, and $(\mathbf{R} \odot \mathbf{S}) \odot \mathbf{Q} = \mathbf{R} \odot (\mathbf{S} \odot \mathbf{Q})$. Since $(\mathbb{C}^{\mathbf{R}}, \circ, \mathbf{0})$ is a BCK-algebra, we denote the algebra that also uses "$\odot$" and specifying the maximal element $\mathbf{1}$, as $(\mathbb{C}^{\mathbf{R}}, \circ, \odot, \mathbf{0}, \mathbf{1})$, and call this an *operator algebra*.

Definition 21 Following Mundici [16], define that if $\mathcal{A} = (\mathcal{B}, +, \bullet, \sim, 0, 1)$ is an algebra of type $\langle 2, 2, 1, 0, 0 \rangle$, satisfying:

[10] The definition of BCK-algebra is from Young and Dudak [22].

(i) $(x+y)+z=x+(y+z)$;
(ii) $x+0=x$;
(iii) $x+y=y+x$;
(iv) $x+1=1$;
(v) $x^{\sim\sim}=x$;
(vi) $0^{\sim}=1$;
(vii) $x+x^{\sim}=1$;
(viii) $(x^{\sim}+y)^{\sim}+y=\sim(x+y^{\sim})+x$;
(ix) $x\bullet y=\sim(x^{\sim}+y^{\sim})$

then $\mathcal{A}$ is an *MV-algebra*.

Theorem 2 *Let* $(\mathbb{C}^{\mathbf{R}}, \circ, \odot, \mathbf{0}, \mathbf{1})$ *be an operator algebra. Let*:

$^{\sim} := \mathbf{R}\mathrm{C}^{\sim} = 1 \circ \mathbf{R}\mathrm{C}$;
$\oplus := \mathbf{R}\mathrm{C} \oplus \mathbf{R}\mathrm{D} = \{\mathrm{E} \mid \mathrm{C}, \mathrm{D}, \mathrm{E} \in \mathbb{C}, \mathrm{R}_{\mathrm{CE}} = 1$ *or* $\mathrm{R}_{\mathrm{DE}} = 1$, *or both* $\mathrm{R}_{\mathrm{CE}} = 1$ *and* $\mathrm{R}_{\mathrm{DE}} = 1\}$;
$\bullet := \mathbf{R}\mathrm{C} \bullet \mathbf{R}\mathrm{D} = \{\mathrm{E} \mid \mathrm{C}, \mathrm{D}, \mathrm{E} \in \mathbb{C}, \mathrm{R}_{\mathrm{CE}} = 1$ *and* $\mathrm{R}_{\mathrm{DE}} = 1\}$, *and let*
$\mathcal{A} := (\mathbb{C}^{\mathbf{R}}, \oplus, \bullet, ^{\sim}, \mathbf{0}, \mathbf{1})$.

Then $\mathcal{A}$ *is an MV-algebra.*

Proof Mundici [16] in Lemma 4 of that paper demonstrated that if $(\mathcal{H}, {}^{*}, 0, 1)$ is a BCK-algebra, then for $x, y \in \mathcal{H}$, if

$$\text{D1: } x^{\sim} = (1^{*}x) \quad \text{and} \quad \text{D2: } x+y = (x^{\sim *}y) \quad \text{and} \quad \text{D3: } x \bullet y = \sim(x^{\sim}+y^{\sim})^{\sim}$$

then $\mathcal{H} = (\mathcal{H}, +, \bullet, ^{\sim}, 0, 1)$ is an MV-algebra. An operator algebra $(\mathbb{C}^{\mathbf{R}}, \circ, \odot, \mathbf{0}, \mathbf{1})$ is a BCK-algebra of the sort required. The values of x and y in the definitions of the objects $\mathbf{R}\mathrm{C} \in \mathbb{C}^{\mathbf{R}}$ are simply the values 0 or 1. Thus, by substitution of the four possible pairs of values into the definition D2 we find that $1+1=1+0=0+1=1$ and $0+0=0$; thus $\mathrm{R}_{\mathrm{CE}} = 1$ or $\mathrm{R}_{\mathrm{DE}} = 1$, or both $\mathrm{R}_{\mathrm{CE}} = 1$ and $\mathrm{R}_{\mathrm{DE}} = 1$; which is the definition of $\oplus$. The pairs of values for D3 are $1 \bullet 1 = 1$, and $1 \bullet 0 = 0 \bullet 1 = 0 \bullet 0 = 0$; that is both $\mathrm{R}_{\mathrm{CE}} = 1$ and $\mathrm{R}_{\mathrm{DE}} = 1$, which is the definition of $\odot$. By substitution of "$\circ$" for "$*$", then $^{\sim}$ is an operation meeting D1, $\oplus$ an operation meeting D2, and $\bullet$ (which is equivalent to $\odot$ of Definition 20) is an operation meeting D3. □

Thus substituting for $\bullet$ with $\odot$ in the MV-algebra, if $(\mathbb{C}^{\mathbf{R}}, \circ, \odot, \mathbf{0}, \mathbf{1})$ is an operator algebra, we can derive the MV-algebra $\mathcal{A} = (\mathbb{C}^{\mathbf{R}}, \oplus, \odot, ^{\sim}, \mathbf{0}, \mathbf{1})$ by Mundici Lemma 4. We next show that the MV operation $\oplus$ is equivalent to sequential application of rule operators, such as the form as **SRC**.[11]

Theorem 3 *Let* $\mathbb{S} = (\mathbb{P}, \mathbb{R}, D, B, M)$ *evolutionary structure with time index set* $\mathcal{T}$ *and descent sequence* Δ. *Let* $\mathbb{A}$ *be the accessibility structure of* $\mathbb{S}$. *Let* $(\mathbb{C}^{\mathbf{R}}, \circ, \odot, \mathbf{0}, \mathbf{1})$ *be an operator algebra with derived MV-algebra* $(\mathbb{C}^{\mathbf{R}}, \oplus, \odot, ^{\sim}, \mathbf{0}, \mathbf{1})$. *Let* $\mathrm{R} \in \mathbb{R}$ *with operator* $\mathbf{R} \in \mathbf{R}$. *Let* $\mathbf{S}\mathbf{R}\mathrm{C} = \bigcup_{\mathrm{D}} \mathrm{SD}$, $\mathrm{D} \in \mathbf{C}\mathrm{R}$. *Then* $\mathbf{SRC} = \mathrm{A}_{\mathrm{RD}} \oplus \mathrm{A}_{\mathrm{RE}} \oplus \mathrm{A}_{\mathrm{RF}}, \oplus \cdots$ *for* $\mathrm{D}, \mathrm{E}, \mathrm{F}, \ldots \in \mathbf{R}\mathrm{C}$. (*That is, the form* **SRC** *is computed by* $\oplus$, *given* **R**C.)

[11] Note, the algebras $(\mathbb{R}, \oplus)$ and $(\mathbf{R}, \oplus)$ are each associative since placement of parenthesis around subsequences does not affect the result.

Proof $\mathrm{F} \in (\mathbf{S}\mathrm{D} \cup \mathbf{S}\mathrm{E})$ iff ($\mathrm{F} \in \mathbf{S}\mathrm{D}$ or $\mathrm{F} \in \mathbf{S}\mathrm{E}$ or both). That is, iff ($\mathrm{S}_{\mathrm{D,F}} = 1$, or $\mathrm{S}_{\mathrm{D,E}} = 1$, or both $\mathrm{S}_{\mathrm{D,E}} = 1$ and $\mathrm{S}_{\mathrm{D,F}} = 1$). That is, $\mathrm{F} \in (\mathbf{S}\mathrm{D} \cup \mathbf{S}\mathrm{E})$ iff $a_{\mathrm{DF}} = 1$ or $a_{\mathrm{EF}} = 1$, or both $a_{\mathrm{DF}} = 1$ and $a_{\mathrm{EF}} = 1$. Now, $\oplus$ implies that $1 \oplus 1 = 1 \oplus 0 = 0 \oplus 1 = 1$ and $0 \oplus 0 = 0$. That is, $\mathrm{F} \in (\mathbf{S}\mathrm{D} \cup \mathbf{S}\mathrm{E})$ iff $a_{\mathrm{SD}} \oplus a_{\mathrm{SE}} = 1$ at position F, and this occurs if either or both $a_{\mathrm{DF}} = 1$, $a_{\mathrm{EF}} = 1$. Thus we can now also write

$$\mathbf{SR}\mathrm{C} = \mathrm{A}_{\mathrm{SD}} \oplus \mathrm{A}_{\mathrm{SE}} \oplus \mathrm{A}_{\mathrm{SF}}, \oplus \cdots \quad \text{for D, E, F}, \ldots \in \mathbf{R}\mathrm{C}. \qquad \square$$

Thus, the two important empirical operations “simultaneous application of two rules” and “sequential application of two rules” are the “natural result” of the MV-algebra $(\mathbb{C}^{\mathbf{R}}, \oplus, \odot, \tilde{}\,, \mathbf{0}, \mathbf{1})$ derived from the operator algebra $(\mathbb{C}^{\mathbf{R}}, \circ, \odot, \mathbf{0}, \mathbf{1})$, built in turn from the GDP $(\mathbb{C}, -, 0)$.

7 Discussion

It has long been the practice of ethnographers to use regular structures to illustrate the structure of a marriage rule and/or the application of a kinship terminology of a culture, without realizing the tremendous analytical power they imply. Part of that power must surely be their relation to the MV-algebra of the rules whose operation they represent. The importance is illustrated by this simple idea:

Definition 22 A rule $\mathrm{R} \in \mathbb{R}$ is *viable* on the descent sequence Δ of the evolutionary structure $\mathbb{S}$ with time index $\mathcal{T}$, iff for all $t \in \mathcal{T}$, $\mathbf{R}\mathrm{C} \neq \emptyset$. If a rule is viable on a descent sequence we also say that the descent sequence is viable, and that the evolutionary structure in which the descent sequence is defined is viable.

A principal purpose of this research is to determine conditions under which a rule, a set of rules, a descent sequence, and the corresponding evolutionary structure, is viable. If we define a “history” as the result of application of any sequence of rules (that is, of rule operators), then the existence of viable histories is a fundamental question of many areas of science.

We also noted that since the descent maps $\mathbf{D}$ are surjections we can derive a density function for that operation. That is significant since this density function then allows computation of the average size of the sibships ($\mathcal{B}$ sets) associated with use of a given rule (“average family size”), and predicts other “demographic” observables. It can be shown [8] that the “eigenstates” of the descent operator (that is, the “fixed points” of the induced map $\mathbf{D} : \mathbb{C} \to \mathbb{C}$) predict distinct possible observable rules of cultural systems, in “pure systems”, and their associated “demographic” statistics, by application of Lemma 1. Linear combinations of pure state characteristics predict statistics of mixed states.[12] This therefore also allows to predict certain effects of cultural change, by use of a commutator of observables that can be found from properties of rules within the algebras studied in this paper. In short, cultural theory has many of the properties of a quantum structure. The implications of this suggestion should be explored.

[12] See Ballonoff [3, 4, 8].

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